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Jeffreys's Correlation Bayes Factor Based on Criteria of Bayarri, Berger, Forte, and Garcia-Donato (2012)

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Main Result:

For bivariate normal data $d = (\bar{x}, \bar{y}, s, t, r)$ the Bayes factor in favour of the alternative $\rho \in (-1, 1)$ against the null $\rho = 0$ is given by:

$$\text{BF}_{10\,;\,\alpha}(n, r) = \frac{2^{1-2\alpha}\sqrt{\pi}}{\mathcal{B}(\alpha, \alpha)} \frac{\Gamma\left(\frac{n+2\alpha-1}{2}\right)}{\Gamma\left(\frac{n+2\alpha}{2}\right)} {}_2F_1\left(\frac{n-1}{2}, \frac{n-1}{2}\,;\, \frac{n+2\alpha}{2}\,;\, r^2\right)$$

Constructed from priors: $\pi(\theta_0) = \pi(\mu_x, \mu_y, \sigma, \tau) = \sigma^{-1}\tau^{-1}$ under the null and $\pi(\theta_0, \rho) = \pi(\theta_0)\pi(\rho\,;\,\alpha)$ under the alternative, where

$$\pi(\rho\,;\,\alpha) = \frac{2^{1-2\alpha}}{\mathcal{B}(\alpha, \alpha)}(1 - \rho^2)^{\alpha-1}$$

The resulting Bayes factor is: Model selection consistent, predictively matched, information consistent when $\alpha \leq 1/2$ with priors that are scale invariant.

Proof:

- Integrate wrt μ_x, μ_y as usual
- Integrate wrt σ, τ using Erdélyi et al. (1954, Eq. 3, p. 313) and Erdélyi et al. (1953, Eq. 4, p. 117) yielding

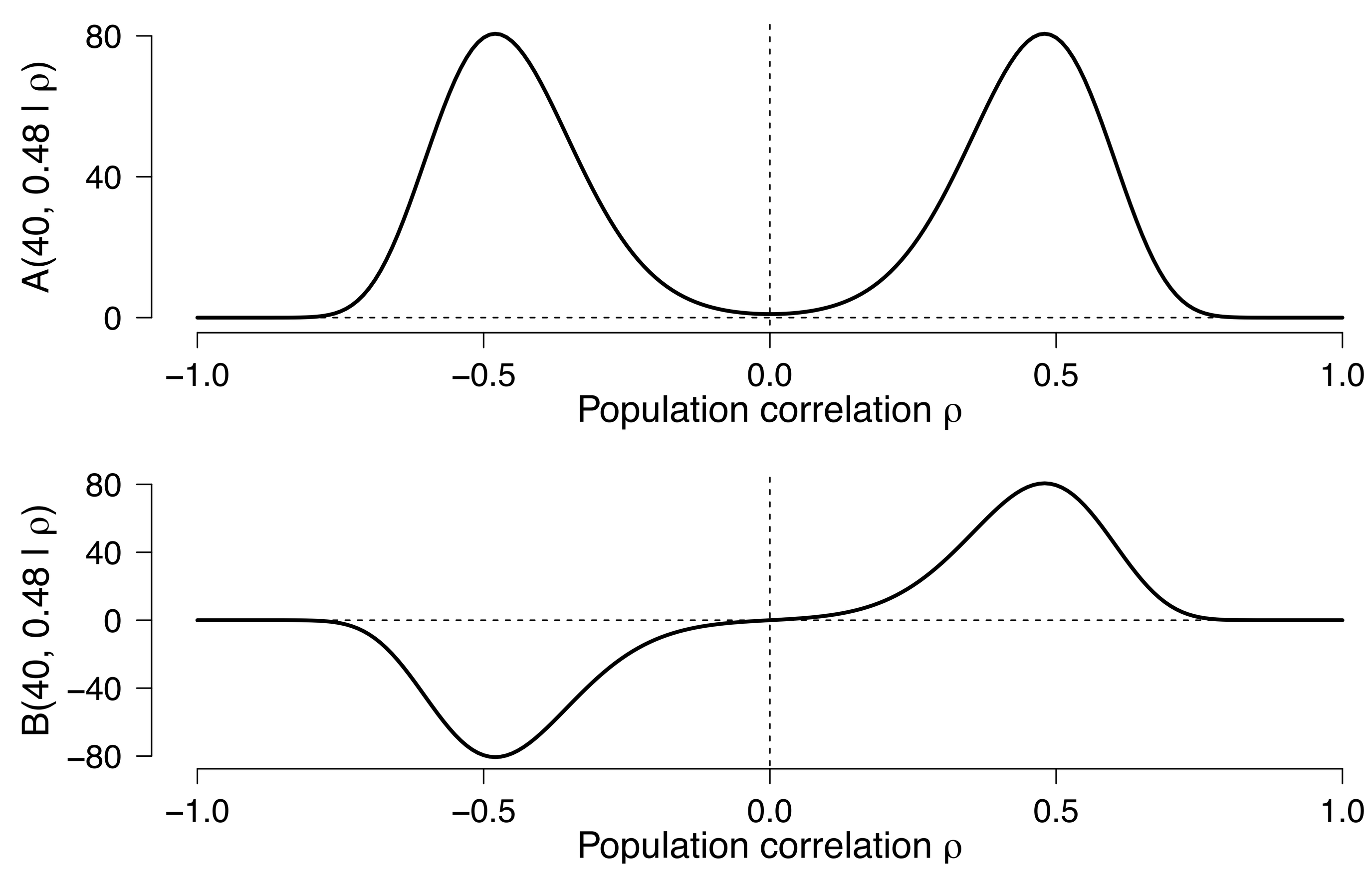
$$\int f(d \,|\, \theta_0 \rho)\pi(\theta_0)\mathrm{d}\theta_0 = p(d \,|\, \mathcal{M}_0) h(n, r \,|\, \rho)$$

where $h(n, r \,|\, \rho) = A(n, r \,|\, \rho) + B(n, r \,|\, \rho)$ and

$$A(n, r \,|\, \rho) = (1 - \rho^2)^{\frac{n-1}{2}} {}_2F_1\left(\tfrac{n-1}{2}, \tfrac{n-1}{2}\,;\, \tfrac{1}{2}\,;\, (r\rho)^2\right)$$

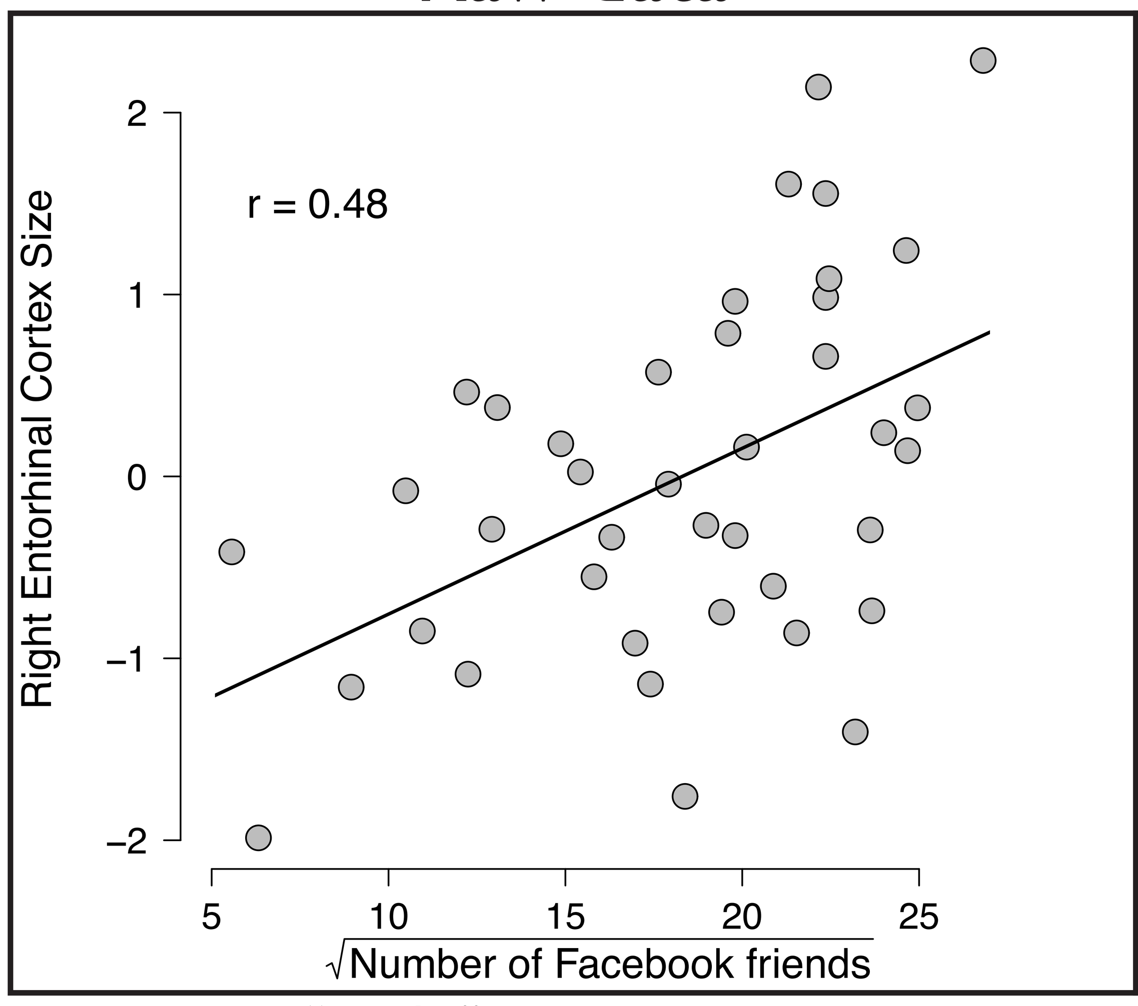
$$B(n, r \,|\, \rho) = 2r\rho(1 - \rho^2)^{\frac{n-1}{2}} \left[\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\right]^2 {}_2F_1\left(\tfrac{n}{2}, \tfrac{n}{2}\,;\, \tfrac{3}{2}\,;\, (r\rho)^2\right)$$

- Integrate ρ using Gradshteyn & Ryzhik (2007, Eq 7.621.4, p 822).



Example: Social network size and brain area.

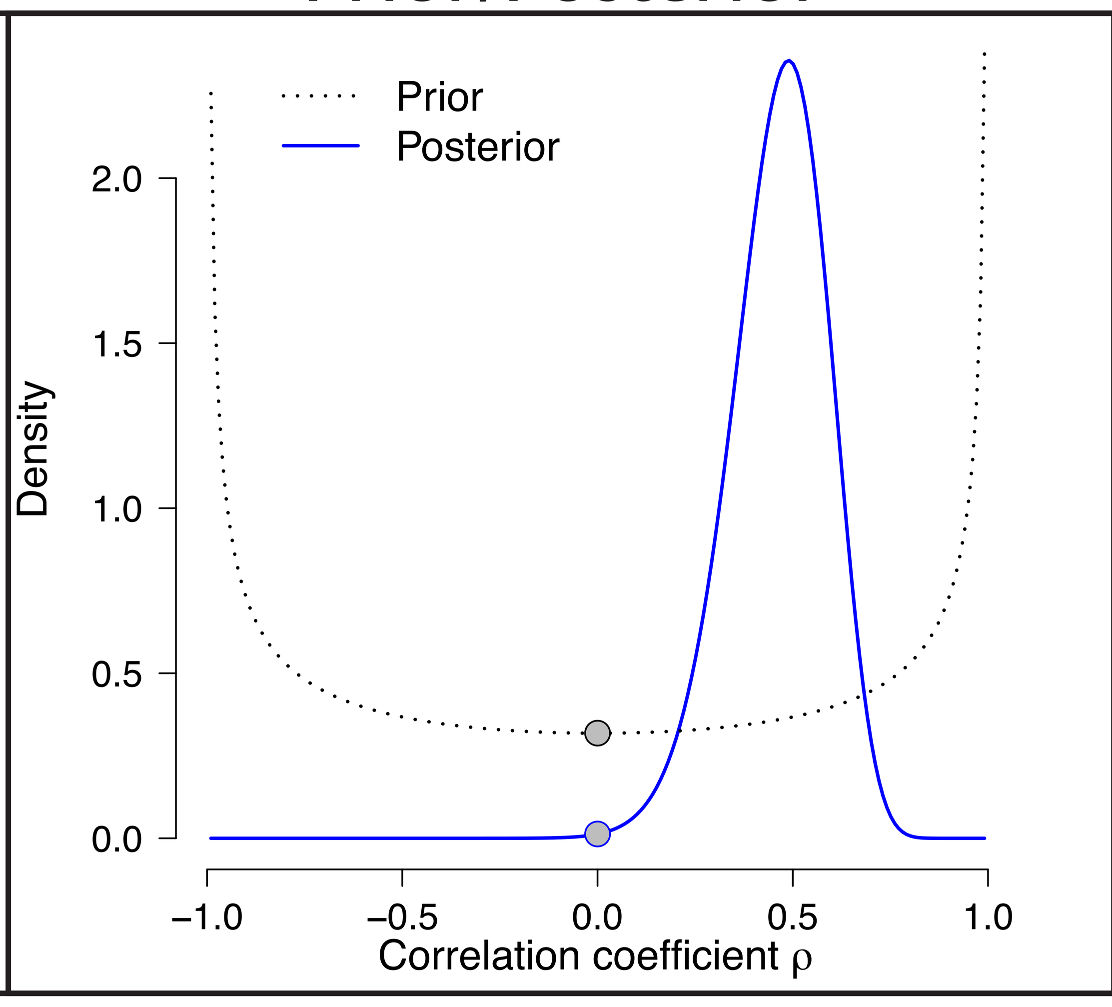
Raw data



Frequentist "results":

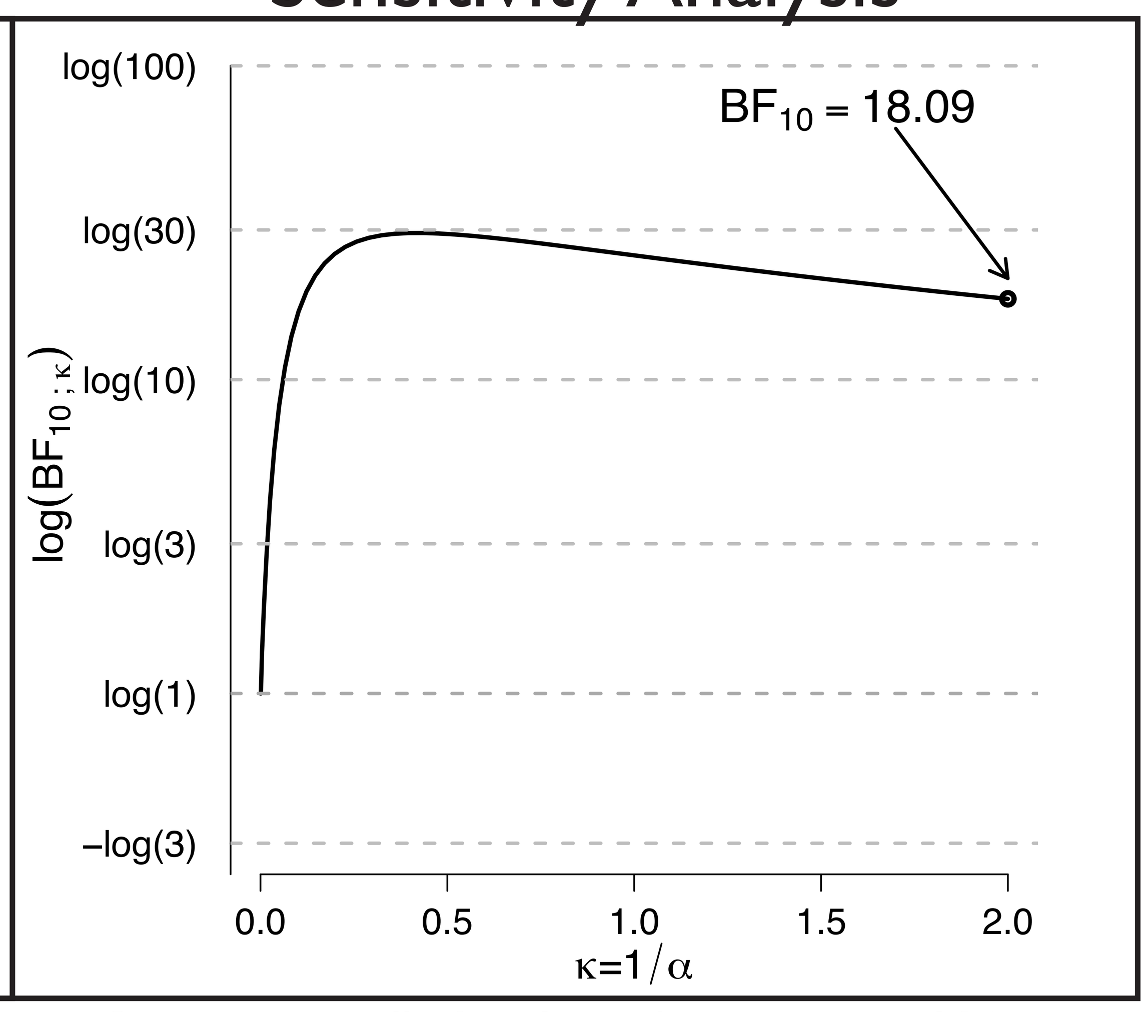
$r(40) = 0.48, p = .00015.$

Prior/Posterior



Prior based on $\alpha = 1/2$, thus, $\kappa = 2$ and the posterior from h and normalisation constant $\text{BF}_{10\,;\,\kappa=2}(n = 40, r = 0.48)$.

Sensitivity Analysis



Computationally fast: It took on average 106 nanoseconds to calculate a thousand Bayes factors.

Thanks to Kanai, R., Bahrami, B., Roylance, R., & Rees, G. (2011). Online social network size is reflected in human brain structure. *Proceedings of the Royal Society B*, 279, 1327-1334.

Corollary: Reference posterior for ρ.

$$\pi_{\text{ref}}(\rho \,|\, n, r) = \frac{\Gamma\left(\frac{n}{2}\right)\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2 {}_2F_1\left(\frac{n-1}{2}, \frac{n-1}{2}\,;\, \frac{1}{2}\,;\, (r\rho)^2\right) + 2r\rho\left[\Gamma\left(\frac{n}{2}\right)\right]^3 {}_2F_1\left(\frac{n}{2}, \frac{n}{2}\,;\, \frac{3}{2}\,;\, (r\rho)^2\right)}{\left[\Gamma\left(\frac{n-1}{2}\right)\right]^3 \sqrt{\pi}\, {}_2F_1\left(\frac{n-1}{2}, \frac{n-1}{2}\,;\, \frac{n}{2}\,;\, r^2\right)} (1 - \rho^2)^{\frac{n-3}{2}}$$

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