# **Time Series**

#### Don van den Bergh

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### Decomposing Time Series

Time series consist of:

- Trend M
- Seasonal S
- Random *R*



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### **Decomposing Time Series**

Time series consist of:

- Trend M
- Seasonal S
- Random *R*

Models are additive or multiplicative:

$$X_t = M_t + S_t + R_t$$
$$X_t = M_t S_t R_t$$

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### Decomposing Time Series



Time Series

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Introduction ••• ••••••• How to analyze the data?

Time Series

Assumptions

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#### Method 1: Trend Estimation

$$\hat{m}_t^q = (2q-1)^{-1} \sum_{j=-q}^q X_{t-j}$$
  
 $\hat{Y}_t = X_t - \hat{m}_t^q$ 

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### Method 1: Trend Estimation q = 2



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### Method 2: Trend Elimination by Differencing

$$abla X_t = X_t - X_{t-1} = (1 - B)X_t$$
 $BX_t = X_{t-1}$ 
 $B^j X_t = X_{t-j}$ 

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### Method 2: Trend Elimination by Differencing

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$$\nabla^{2}X_{t} = \nabla (\nabla X_{t}) = X_{t} - 2X_{t-1} - X_{t-2}$$

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Imagine:

$$m_t = c_0 + c_1 t$$

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Imagine:

$$m_t = c_0 + c_1 t$$
  
 $\nabla m_t = m_t - m_{t-1} = c_0 + c_1 t - (c_0 + c_1 (t-1))$   
 $\nabla m_t = c_1$ 

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### Method 2: Trend Elimination by Differencing

This generalizes:

$$egin{aligned} X_t &= m_t + Y_t & Y_t ext{ is stationary with } \mu_Y(t) = 0 \ m_t &= \sum_{j=0}^k c_j t^j \ 
abla^k X_t &= k! c_k + 
abla^k Y_t \end{aligned}$$

Now,  $\nabla^k X_t$  is stationary with mean  $k!c_k$ .

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### Method 2: Trend Elimination by Differencing q = 2



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### Summarizing the preprocessing step

- Visualize the time series
  - Detect Trend and Seasonal effects
- Transform data such that residuals are stationary
  - Estimate and substract  $M_t$  and  $S_t$
  - Differencing
  - Transformations (log,  $\sqrt{}$ )
- Fit time series model to the stationary residuals

Introduction 00 000000 Stationarity 

### Stationarity

Some notation:

$$\mu_X(t) = E[X_t]$$
  

$$\gamma_X(s, t) = \text{Cov}(X_s, X_t) = E[(X_s - \mu_s)(X_t - \mu_t)]$$



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# Stationarity

Strict:

$$(X_1,\ldots,X_n) \stackrel{d}{=} (X_{1+h},\ldots,X_{n+h})$$

Weak:

- $\mu_X(t)$  is independent of t
- $\gamma_X(t+h,t)$  is independent of t for each h.

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# Stationarity

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- $\gamma_X(t+h,t)$  is independent of t for each h.

Strict usually implies weak but not the other way around.

# Stationarity: Random Walk

$$S_t = \sum_{i=1}^t X_i$$
 where  $X_i$  i.i.d.,  $\mu_X = 0$  and  $\gamma_X = \sigma^2$   
 $E[S_t] = 0$  and  $E[S_t^2] = t\sigma^2$ . Hence:

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# Stationarity: Random Walk

$$S_t = \sum_{i=1}^t X_i$$
 where  $X_i$  i.i.d.,  $\mu_X = 0$  and  $\gamma_X = \sigma^2 E[S_t] = 0$  and  $E[S_t^2] = t\sigma^2$ . Hence:

$$\gamma_{\mathcal{S}}(t+h,t) = \operatorname{Cov}(S_{t+h},S_t)$$
$$= t\sigma^2 - 0^2$$

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### Stationarity: Random Walk

$$S_t = \sum_{i=1}^t X_i$$
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 $E[S_t] = 0$  and  $E[S_t^2] = t\sigma^2$ . Hence:  
 $\gamma_S(t+h,t) = \text{Cov}(S_{t+h}, S_t)$ 

$$= t\sigma^2 - 0^2$$

We conclude: Random walks violate stationarity.

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### Linear process

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}$$

where 
$$W_t \sim \mathcal{N}\left(0, \, \sigma^2
ight)$$
, and  $\sum_{j=-\infty}^\infty |\psi_j| < \infty$ 



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#### Linear process

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, and  $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ 

#### Wold's decomposition

Every second-order stationary process is, or can be transformed to, a linear process.

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Assumptions

MA(1)

$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}$$

Choose  $\mu = 0$ ,

$$\psi_j = egin{cases} 1, & ext{if } j = 0 \ heta, & ext{if } j = 1 \ 0, & ext{otherwise} \end{cases}$$

we obtain:

$$X_t = W_t + \theta W_{t-1}$$

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### Properties of the MA(1)

MA(1):

$$Var(X_t) = \gamma_X(t, t) = \sigma^2(1 + \theta^2)$$
$$Cov(X_t) = \begin{cases} \gamma_X(t, t+h) = \theta\sigma^2, \text{ if } h = 1\\ \gamma_X(t, t+h) = 0, \text{ if } h > 1 \end{cases}$$
$$Cor_{X_t}(t, t+h) = \frac{\gamma_X(t, t+h)}{\gamma_X(t, t)} = \begin{cases} \frac{\theta}{1+\theta^2}, \text{ if } h = 1\\ 0, \text{ if } h > 1 \end{cases}$$

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Assumptions



$$X_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j W_{t-j}$$

Choose  $\mu = 0$ ,

$$\psi_j = egin{cases} 
ho^j, & ext{if } j \geq 0 \ 0, & ext{otherwise} \end{cases}$$

Then if  $|\rho| < 1$  we obtain:

$$X_t = \rho X_{t-1} + W_t$$

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Assumptions

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### Properties of the AR(1)

AR(1):

$$\begin{aligned} \mathsf{Var}(X_t) &= \gamma_X(t,t) = \frac{\sigma^2}{(1-\rho^2)}\\ \mathsf{Cov}(X_t) &= \gamma_X(t,t+h) = \frac{\rho^{|h|}\sigma^2}{(1-\rho^2)}\\ \mathsf{Cor}_{X_t}(t,t+h) &= \frac{\gamma_X(t,t+h)}{\gamma_X(t,t)} = \rho^{|h|} \end{aligned}$$

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# Comparison of MA(1) and AR(1)

- The difference is in the correlation and covariance function
- Useful for estimation
- Useful for model comparison

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# Comparison of MA(1) and AR(1): ACF



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Assumptions

### Extensions: AR(p)

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = W_t$$

where  $W_t \sim \mathcal{N}(0, \sigma^2)$ 



Assumptions

### Extensions: AR(p)

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = W_t$$

where  $W_t \sim \mathcal{N}(0,\sigma^2)$ 

Equivalently,

$$\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$$
  
$$\phi(B) X_t = W_t$$

Remember:  $BX_t = X_{t-1}$  and  $B^j X_t = X_{t-j}$ 

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### Extensions: AR(p)

Conditions on  $\phi(B)$ 

$$\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$$

- Unique solution iff no roots in the unit circle
- Does not depend on future iff roots outside of unit circle

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### Extensions: ARMA(p, q)

ARMA(1, 1)

$$X_t - \phi X_{t-1} = W_t + \theta W_{t-1}$$

where  $\phi + \theta \neq 0$ 



# Extensions: ARMA(p, q)

ARMA(1, 1)

$$X_t - \phi X_{t-1} = W_t + \theta W_{t-1}$$

where  $\phi + \theta \neq 0$ 

ARMA(p, q)

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \dots + \theta_q W_{t-q}$$
$$\phi(B) X_t = \theta(B) W_t$$

For identifiability,  $\phi(B)$  and  $\theta(B)$  have no common factors.

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#### Extensions: predictors

- VAR model
- VMA model
- VARMA model
- R: stats::arima and forecasting

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